Induced Two-Crossed Modules

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Abstract

We introduce the notion of an induced 2-crossed module, which extends the notion of an induced crossed module (Brown and Higgins).

Introduction

Induced crossed modules were defined by Brown and Higgins [2] and studied further in paper by Brown and Wensley [4, 5]. This is looked at in detail in a book by Brown, Higgins and Sivera [3]. Induced crossed modules allow detailed computations of non-Abelian information on second relative groups.

To obtain analogous result in dimension 2, we make essential use of a 2-crossed module defined by Conduché [6].

A major aim of this paper is to introduce induced 2-crossed modules

$$\{\phi_*(L), \phi_*(M), Q, \partial_2, \partial_1\}$$

which can be used in applications of the 3-dimensional Van Kampen Theorem.

The method of Brown and Higgins [2] is generalized to give results on $\{\phi_*(L), \phi_*(M), Q, \partial_2, \partial_1\}$. However; Brown, Higgins and Sivera [3] indicate a bifibration from crossed squares, so leading to the notion of induced crossed square, which is relevant to a triadic Hurewicz theorem in dimension 3.

1 Preliminaries

Throughout this paper all actions will be left. The right actions in some references will be rewrite by using left actions.

1.1 Crossed Modules

Crossed modules of groups were initially defined by Whitehead [11, 12] as models for (homotopy) 2-types. We recall from [9] the definition of crossed modules of groups.

A crossed module, (M, P, ∂), consists of groups M and P with a left action of P on M, written $(p, m) \mapsto {}^{p}m$ and a group homomorphism $\partial : M \to P$ satisfying the following conditions:

$$CM1) \qquad \partial({}^{p}m) = p\partial(m)p^{-1} \qquad \text{and} \qquad CM2) \qquad {}^{\partial(m)}n = mnm^{-1}$$

for $p \in P, m, n \in M$. We say that $\partial : M \to P$ is a pre-crossed module, if it is satisfies CM1.

If (M, P, ∂) and (M', P', ∂') are crossed modules, a morphism,

$$(\mu, \eta) : (M, P, \partial) \to (M', P', \partial'),$$

of crossed modules consists of group homomorphisms $\mu : M \to M'$ and $\eta : P \to P'$ such that

$$(i)\ \eta\partial = \partial'\mu \qquad \text{and} \qquad (ii)\ \mu\left({}^{p}m\right) = {}^{\eta(p)}\mu\left(m\right)$$

for all $p \in P, m \in M$.

Crossed modules and their morphisms form a category, of course. It will usually be denoted by XMod. We also get obviously a category PXMod of pre-crossed modules.

There is, for a fixed group P, a subcategory XMod/P of XMod, which has as objects those crossed modules with P as the "base", i.e., all (M, P, ∂) for this fixed P, and having as morphisms from (M, P, ∂) to (M', P', ∂') those (μ, η) in XMod in which $\eta : P \to P'$ is the identity homomorphism on P.

Some standart examples of crossed modules are:

(i) normal subgroup crossed modules $(i : N \to P)$ where i is an inclusion of a normal subgroup, and the action is given by conjugation;

(ii) automorphism crossed modules $(\chi : M \to Aut(M))$ in which

$$(\chi m)(n) = mnm^{-1};$$

(iii) Abelian crossed modules $1 : M \to P$ where M is a P-module;

(iv) central extension crossed modules $\partial : M \to P$ where ∂ is an epimorphism with kernel contained in the center of M.

Induced crossed modules were defined by Brown and Higgins in [2] and studied further in papers by Brown and Wensley [4, 5].

We recall from [3] below a presentation of the induced crossed module which is helpful for the calculation of colimits.

1.2 Pullback Crossed Modules

Definition 1 *Let $\phi : P \to Q$ be a homomorphism of groups and let $\mathcal{N} = (N, Q, v)$ be a crossed module. We define a subgroup*

$$\phi^*(N) = N \times_Q P = \{(n, p) \mid v(n) = \phi(p)\}$$

of the product $N \times P$. This is usually pullback in the category of groups. There is a commutative diagram

$$\begin{tikzcd} \phi^*(N) \arrow[r, "\bar{\phi}"] \arrow[d, "\bar{v}"'] & N \arrow[d, "v"] \\ P \arrow[r, "\phi"'] & Q \end{tikzcd}$$

where $\bar{v} : (n,p) \mapsto p$, $\bar{\phi} : (n,p) \mapsto n$. *Then* P *acts on* $\phi^*(N)$ *via* ϕ *and the diagonal, i.e.* ${}^{p'}(n,p) = ({}^{\phi(p')}n, p'pp'^{-1})$. *It is easy to see that this gives a p-action. Since*

$$\begin{aligned}(n,p)(n_1,p_1)(n,p)^{-1} &= \left(nn_1n^{-1}, pp_1p^{-1}\right)\\ &= \left({}^{v(n)}n_1, pp_1p^{-1}\right)\\ &= \left({}^{\phi(p)}n_1, pp_1p^{-1}\right)\\ &= {}^{\bar{v}(n,p)}(n_1,p_1),\end{aligned}$$

we get a crossed module $\phi^*(\mathcal{N}) = (\phi^*(N), P, \bar{v})$ *which is called the pullback crossed module of* $\mathcal{N}$ *along* ϕ. *This construction satisfies a universal property, analogous to that of the pullback of groups. To state it, we use also the morphism of crossed modules*

$$\left(\bar{\phi}, \phi\right) : \phi^*(\mathcal{N}) \to \mathcal{N}.$$

Theorem 2 *For any crossed module* $\mathcal{M} = (M, P, \mu)$ *and any morphism of crossed modules*

$$(h, \phi) : \mathcal{M} \to \mathcal{N}$$

there is a unique morphism of crossed P*-modules* $h' : \mathcal{M} \to \phi^*(\mathcal{N})$ *such that the following diagram commutes*

$$\begin{tikzcd}
M \arrow[dr, dashed, "h'"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "\mu"'] & & \\
& \phi^*(N) \arrow[r, "\bar{\phi}"'] \arrow[d, "\bar{v}"] & N \arrow[d, "v"'] \\
& P \arrow[r, "\phi"] & Q.
\end{tikzcd}$$

This can be expressed functorially:

$$\phi^* \colon \mathsf{XMod}/Q \ \to \mathsf{XMod}/P$$

which is a pullback functor. This functor has a left adjoint

$$\phi_* \colon \mathsf{XMod}/P \ \to \mathsf{XMod}/Q$$

which gives a induced crossed module as follows.

1.3 Induced Crossed Modules

Definition 3 *For any crossed* P*-module* $\mathcal{M} = (M, P, \mu)$ *and any homomorphism* $\phi : P \to Q$ *the crossed module induced by* ϕ *from* μ *should be given by:*

(i) *a crossed* Q*-module* $\phi_*(\mathcal{M}) = (\phi_*(M), Q, \phi_*\mu)$,

(ii) *a morphism of crossed modules* $(f, \phi) : \mathcal{M} \to \phi_*(\mathcal{M})$, *satisfying the dual universal property that for any morphism of crossed modules*

$$(h, \phi) : \mathcal{M} \to \mathcal{N}$$

there is a unique morphism of crossed Q-modules $h' : \phi_*(M) \to N$ such that the diagram

$$\begin{tikzcd}
 & & N \arrow[dl, bend left, "v"] \\
 & & \\
M \arrow[r, "f"'] \arrow[d, "\mu"] \arrow[uurr, bend left, "h"] & \phi_*(M) \arrow[d, "\phi_*\mu"'] \arrow[uur, dashed, "h'"] & \\
P \arrow[r, "\phi"] & Q &
\end{tikzcd}$$

commutes.

Now we briefly explain this from Brown and Higgins, [2] as follows, (see also [1]).

Proposition 4 *Let $\mu : M \to P$ be a crossed P-module and let $\phi : P \to Q$ be a morphism of groups. Then the induced crossed Q-module $\phi_*(M)$ is generated, as a group, by the set $M \times Q$ with defining relations*

$$\begin{array}{ll} (i) & (m_1, q)(m_2, q) = (m_1 m_2, q), \\ (ii) & ({}^p m, q) = (m, q\phi(p)), \\ (iii) & (m_1, q_1)(m_2, q_2)(m_1, q_1)^{-1} = (m_2, q_1 \phi\mu(m_1) q_1^{-1} q_2) \end{array}$$

for $m, m_1, m_2 \in M$, $q, q_1, q_2 \in Q$ and $p \in P$.

The morphism $\phi_*\mu : \phi_*(M) \to Q$ is given by $\phi_*\mu(m, q) = q\phi\mu(m)q^{-1}$, the action of Q on $\phi_*(M)$ by ${}^q(m, q_1) = (m, qq_1)$, and the canonical morphism $\phi' : M \to \phi_*(M)$ by $\phi'(m) = (m, 1)$.

The crossed module $(\phi_*(M), Q, \phi_*\mu)$, thus defined in Proposition 4, is called the *induced crossed module* of (M, P, μ) along ϕ.

If $\phi : P \to Q$ is an epimorphism the induced crossed module $(\phi_*(M), Q, \phi_*\mu)$ has a simpler description.

Proposition 5 ([*2*], *Proposition* 9) *If $\phi : P \to Q$ is an epimorphism, and $\mu : M \to P$ is a crossed module, then $\phi_*(M) \cong M/[K, M]$, where $K = Ker\phi$, and $[K, M]$ denotes the subgroup of M generated by all ${}^k m m^{-1}$ for all $m \in M, k \in K$.*

2 Two-Crossed Modules

Conduché [6] described the notion of a 2-crossed module as a model of connected homotopy 3-types.

A 2-*crossed module* is a normal complex of groups $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$ together with an action of P on all three groups and a mapping

$$\{-,-\} : M \times M \to L$$

which is often called the Peiffer lifting such that the action of P on itself is by conjugation, ∂_2 and ∂_1 are P-equivariant.

$$
\begin{array}{llll}
\textbf{PL1}: & \partial_2 \{m_0, m_1\} & = & m_0 m_1 m_0^{-1} \left({}^{\partial_1 m_0} m_1^{-1}\right) \\
\textbf{PL2}: & \{\partial_2 l_0, \partial_2 l_1\} & = & [l_0, l_1] \\
\textbf{PL3}: & \{m_0, m_1 m_2\} & = & {}^{m_0 m_1 m_0^{-1}} \{m_0, m_2\} \{m_0, m_1\} \\
 & \{m_0 m_1, m_2\} & = & \left\{m_0, m_1 m_2 m_1^{-1}\right\} \left({}^{\partial_1 m_0} \{m_1, m_2\}\right) \\
\textbf{PL4}: & a)\, \{\partial_2 l, m\} & = & l \left({}^{m} l^{-1}\right) \\
 & b)\, \{m, \partial_2 l\} & = & {}^{m} l \left({}^{\partial_1 m} l^{-1}\right) \\
\textbf{PL5}: & {}^{p} \{m_0, m_1\} & = & \{{}^{p} m_0, {}^{p} m_1\}
\end{array}
$$

for all $m, m_0, m_1, m_2 \in M, l, l_0, l_1 \in L$ and $p \in P$. Note that we have not specified that M acts on L. We could have done that as follows: if $m \in M$ and $l \in L$, define

$$ {}^{m} l = l \left\{\partial_2 l^{-1}, m\right\}. $$

From this equation (L, M, ∂_2) becomes a crossed module.

We denote such a 2-crossed module of groups by $\{L, M, P, \partial_2, \partial_1\}$.

A morphism of 2-crossed modules is given by a diagram

$$
\begin{tikzcd}
L \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & M \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & P \arrow[d, "f_0"'] \\
L' \arrow[r, "\partial_2'"'] & M' \arrow[r, "\partial_1'"'] & P'
\end{tikzcd}
$$

where $f_0 \partial_1 = \partial_1' f_1$, $f_1 \partial_2 = \partial_2' f_2$

$$ f_1\left({}^{p} m\right) = {}^{f_0(p)} f_1(m) \quad , \quad f_2\left({}^{p} l\right) = {}^{f_0(p)} f_2(l) $$

and

$$ \{-,-\}(f_1 \times f_1) = f_2 \{-,-\} $$

for all $m \in M, l \in L$ and $p \in P$.

These compose in an obvious way giving a category which we will denote by $\mathsf{X_2Mod}$. There is, for a fixed group P, a subcategory $\mathsf{X_2Mod}/P$ of $\mathsf{X_2Mod}$ which has as objects those crossed modules with P as the "base", i.e., all $\{L, M, P, \partial_2, \partial_1\}$ for this fixed P, and having as morphism from $\{L, M, P, \partial_2, \partial_1\}$ to $\{L', M', P', \partial_2', \partial_1'\}$ those (f_2, f_1, f_0) in $\mathsf{X_2Mod}$ in which $f_0 : P \to P'$ is the identity homomorphism on P.

Some remarks on Peiffer lifting of 2-crossed modules given by Porter in [9] are:
Suppose we have a 2-crossed module

$$ L \stackrel{\partial_2}{\to} M \stackrel{\partial_1}{\to} P, $$

with extra condition that $\{m, m'\} = 1$ for all $m, m' \in M$. The obvious thing to do is to see what each of the defining properties of a 2-crossed module give in this case.

(i) There is an action of P on L and M and the ∂s are P-equivariant. (This gives nothing new in our special case.)

(ii) $\{-,-\}$ is a lifting of the Peiffer commutator so if $\{m,m'\}=1$, the Peiffer identity holds for (M,P,∂_1), i.e. that is a crossed module;

(iii) if $l,l'\in L$, then $1=\{\partial_2 l,\partial_2 l'\}=[l,l']$, so L is Abelian and,

(iv) as $\{-,-\}$ is trivial ${}^{\partial_1 m}l^{-1}=l^{-1}$, so ∂M has trivial action on L. Axioms PL3 and PL5 vanish.

Examples of 2-Crossed Modules

1. Let $M\overset{\partial}{\to}P$ be a pre-crossed module. Consider the Peiffer subgroup $\langle M,M\rangle\subset M$, generated by the Peiffer commutators

$$\langle m,m'\rangle = mm'^{-1}m^{-1}\left({}^{\partial_1 m}m'\right)$$

for all $m,m'\in M$. Then

$$\langle M,M\rangle \overset{\partial_2}{\to} M \overset{\partial_1}{\to} P$$

is a 2-crossed module with the Peiffer lifting $\{m,m'\}=\langle m,m'\rangle$, [10].

2. Any crossed module gives a 2-crossed module. Given (M,P,∂) is a crossed module, the resulting sequence

$$L\to M\to P$$

is a 2-crossed module by taking $L=1$. This is functorial and XMod can be considered to be a full category of $\mathsf{X_2Mod}$ in this way. It is a reflective subcategory since there is a reflection functor obtained as follows:

If $L\overset{\partial_2}{\to}M\overset{\partial_1}{\to}P$ is a 2-crossed module, then $\mathrm{Im}\partial_2$ is a normal subgroup of M and there is an induced crossed module structure on $\partial_1:\dfrac{M}{\mathrm{Im}\partial_2}\to P$, (c.f. [9]).

Another way of encoding 3-types is using the noting of a crossed square by Guin-Waléry and Loday, [8] .

Definition 6 *A crossed square is a commutative diagram of group morphisms*

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "u"'] & M \arrow[d, "v"] \\ N \arrow[r, "g"'] & P \end{tikzcd}$$

with action of P on every other group and a function $h:M\times N\to L$ such that

(1) the maps f and u are P-equivariant and g, v, $v\circ f$ and $g\circ u$ are crossed modules,

(2) $f\circ h(x,y)=x^{g(y)}x^{-1}$, $u\circ h(x,y)={}^{v(x)}yy^{-1}$,

(3) $h(f(z),y)=z^{g(y)}z^{-1}$, $h(x,u(z))={}^{v(x)}zz^{-1}$,

(4) $h(xx',y)={}^{v(x)}h(x',y)h(x,y)$, $h(x,yy')=h(x,y)^{g(y)}h(x,y')$,

(5) $h({}^t x,{}^t y)={}^t h(x,y)$

for $x, x' \in M$, $y, y' \in N$, $z \in L$ *and* $t \in P$.

It is a consequence of the definition that $f : L \to M$ and $u : L \to N$ are crossed modules where M and N act on L via their images in P. A crossed square can be seen as a crossed module in the category of crossed modules.

Also, it can be considered as a complex of crossed modules of length one and thus, Conduché [7], gave a direct proof from crossed squares to 2-crossed modules. This construction is the following:

Let

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "u"'] & M \arrow[d, "v"] \\ N \arrow[r, "g"'] & P \end{tikzcd}$$

be a crossed square. Then seeing the horizontal morphisms as a complex of crossed modules, the mapping cone of this square is a 2-crossed module

$$L \overset{\partial_2}{\to} M \rtimes N \overset{\partial_1}{\to} P,$$

where $\partial_2(z) = (f(z)^{-1}, u(z))$ for $z \in L, \partial_1(x, y) = v(x)g(y)$ for all $x \in M$ and $y \in N$, and the Peiffer lifting is given by

$$\{(x, y), (x', y')\} = h(x, yy'y^{-1}).$$

Of course, the construction of 2-crossed modules from crossed squares gives a generic family of examples.

3 Pullback Two-Crossed Modules

In this section we introduce the notion of a pullback 2-crossed module, which extends a pullback crossed module defined by Brown-Higgins, [2]. The importance of the "pullback" is that it enables us to move from crossed Q-module to crossed P-module, when a morphism of groups $\phi : P \to Q$ is given.

Definition 7 *Given a* 2*-crossed module* $\{H, N, Q, \partial_2, \partial_1\}$ *and a morphism of groups* $\phi : P \to Q$*, the pullback* 2*-crossed module can be given by*

(i) a 2-crossed module $\phi^* \{H, N, Q, \partial_2, \partial_1\} = \{\phi^*(H), \phi^*(N), P, \partial_2^*, \partial_1^*\}$
(ii) given any morphism of 2-crossed modules

$$(f_2, f_1, \phi) : \{B_2, B_1, P, \partial_2', \partial_1'\} \to \{H, N, Q, \partial_2, \partial_1\},$$

there is a unique (f_2^*, f_1^*, id_P) 2-crossed module morphism that commutes the following diagram:

$$\begin{tikzcd} & (B_2, B_1, P, \partial_2', \partial_1') \arrow[dl, dashed, "(f_2^*, f_1^*, id_P)"'] \arrow[d, "(f_2, f_1, \phi)"] \\ (\phi^*(H), \phi^*(N), P, \partial_2^*, \partial_1^*) \arrow[r, "(id_H, \phi', \phi)"'] & (H, N, Q, \partial_2, \partial_1) \end{tikzcd}$$

or more simply as

$$\begin{tikzcd}
B_2 \arrow[dd, "\partial_2{}'"'] \arrow[dr, dashed, "f_2^*"'] \arrow[drrr, bend left=10, "f_2"] & & & \\
& \phi^*(H) \arrow[rr, equal, "id_H"'] \arrow[dd, "\partial_2^*"] & & H \arrow[dd, "\partial_2"] \\
B_1 \arrow[dd, "\partial_1{}'"'] \arrow[dr, dashed, "f_1^*"'] \arrow[drrr, bend left=10, "f_1"] & & & \\
& \phi^*(N) \arrow[rr, "\phi'"'] \arrow[dd, "\partial_1^*"] & & N \arrow[dd, "\partial_1"] \\
P \arrow[dr, equal, "id_P"'] \arrow[drrr, bend left=10, "\phi"] & & & \\
& P \arrow[rr, "\phi"'] & & Q.
\end{tikzcd}$$

Now, we will construct pullback 2-crossed module. Given are any 2-crossed module $H \overset{\partial_2}{\to} N \overset{\partial_1}{\to} Q$ and any group morphism $\phi : P \to Q$. We can take $\phi^*(N) \overset{\partial_1^*}{\to} P$ as pullback pre-crossed module of $N \overset{\partial_1}{\to} Q$ by ϕ given in definition 1. Then we will define $\phi^*(H) \overset{\partial_2^*}{\to} \phi^*(N)$, $\partial_2^*(h,(n,p)) = (\partial_2 h, p)$ as pullback of ∂_1 by $\phi' : \phi^*(N) \to N$ where

$$\begin{aligned}
\phi^*(H) &= \{(h,(n,p)) \mid \partial_2(h) = \phi'(n,p) = n, \phi(p) = \partial_1(n)\} \\
&= \{(h,(\partial_2(h),p)) \mid \phi(p) = \partial_1(n) = \partial_1(\partial_2(h)) = 1\} \\
&\cong H \times_N (Ker\partial_1 \times \mathrm{Ker}\phi).
\end{aligned}$$

Since pullback of a pullback is a pullback, we get pullback composition

$$\phi^*(H) \overset{\partial_2^*}{\to} \phi^*(N) \overset{\partial_1^*}{\to} P$$

of $\partial_1\partial_2 = 1$ by ϕ. On the other hand, we can construct directly the pullback of $\partial_1\partial_2 = 1$ by ϕ as $\partial : B \to P$ where $B = \{(h,p) \mid \phi(p) = 1\} \cong H \times \ker\phi$. We can define the isomorphism $\Psi : \phi^*(H) \to B$, $\Psi(x) = (h,p)$ where $x = (h,(\partial_2(h),p)) \in \phi^*(H)$. So $\phi^*(H) \cong B$.

We find that the pullback composition $\phi^*(H) \overset{\partial_2^*}{\to} \phi^*(N) \overset{\partial_1^*}{\to} P$ is not normal complex of groups unless ϕ is a monomorphism. To see this, note that for $(h,p) \in H \times \mathrm{Ker}\phi$,

$$\partial_1^*\partial_2^*(h,p) = \partial_1^*(\partial_2 h, p) = p.$$

This last expression is equal to 1 if ϕ is a monomorphism. So $\phi^*(H) \cong H$. Thus, if ϕ is a monomorphism, then we may consider $\{\phi^*(H), \phi^*(N), P, \partial_2^*, \partial_1^*\}$ as a 2-crossed module with the following Peiffer lifting

$$\{-,-\} : \phi^*(N) \times \phi^*(N) \to H$$

given by $\{(n,p),(n',p')\}=\{n,n'\}$.

Proposition 8 *If $H\overset{\partial_2}{\to}N\overset{\partial_1}{\to}Q$ is a 2-crossed module and if $\phi:P\to Q$ is a monomorphism of groups then*

$$H\overset{\partial_2^*}{\to}\phi^*(N)\overset{\partial_1^*}{\to}P$$

is a pullback 2-crossed module where $\partial_2^*(h)=(\partial_2(h),1)$, $\partial_1^*(n,p)=p$, the action of P on $\phi^*(N)$ and H by ${}^p(n,p')=\left({}^{\phi(p)}n,pp'p^{-1}\right)$ and ${}^ph={}^{\phi(p)}h$ respectively.

Proof. ∂_2^* is P-equivariant with the action ${}^p(n,p')=\left({}^{\phi(p)}n,pp'p^{-1}\right)$.

$$\begin{aligned}
{}^p\partial_2^*(h) &= {}^p(\partial_2(h),1)\\
&= ({}^{\phi(p)}\partial_2(h),p1p^{-1})\\
&= ({}^{\phi(p)}\partial_2(h),1)\\
&= (\partial_2\left({}^{\phi(p)}h\right),1)\\
&= (\partial_2({}^ph),1)\\
&= \partial_2^*({}^ph)
\end{aligned}$$

It is clear that ∂_1^* is P-equivariant. The Peiffer lifting

$$\{-,-\}:\phi^*(N)\times\phi^*(N)\to H$$

is given by $\{(n,p),(n',p')\}=\{n,n'\}$.

PL1:

$$\begin{aligned}
& (n,p)(n',p')(n,p)^{-1}\left({}^{\partial_1^*(n,p)}(n',p')^{-1}\right)\\
=\ & (n,p),(n',p')\left(n^{-1},p^{-1}\right)^p\left(n'^{-1},p'^{-1}\right)\\
=\ & (n,p),(n',p')\left(n^{-1},p^{-1}\right)\left({}^{\phi(p)}n'^{-1},pp'^{-1}p^{-1}\right)\\
=\ & \left(nn'n^{-1},pp'p^{-1}\right)\left({}^{\partial_1(n)}n'^{-1},pp'^{-1}p^{-1}\right)\\
=\ & \left(nn'n^{-1}\left({}^{\partial_1(n)}n'^{-1}\right),pp'p^{-1}pp'^{-1}p^{-1}\right)\\
=\ & \left(nn'n^{-1}\left({}^{\partial_1(n)}n'^{-1}\right),1\right)\\
=\ & (\partial_2\{n,n'\},1)\\
=\ & \partial_2^*\{n,n'\}\\
=\ & \partial_2^*\{(n,p),(n',p')\}.
\end{aligned}$$

PL2:

$$\begin{aligned}
\{\partial_2^*h,\partial_2^*h'\} &= \{(\partial_2h,1),(\partial_2h',1)\}\\
&= \{\partial_2h,\partial_2h'\}\\
&= [h,h'].
\end{aligned}$$

The rest of axioms of 2-crossed module is given in appendix.

(ii)

$$(id_H,\phi',\phi):\{H,\phi^*(N),P,\partial_2^*,\partial_1^*\}\to\{H,N,Q,\partial_2,\partial_1\}$$

or diagrammatically,

$$\begin{tikzcd}
H \arrow[r, equal, "id_H"] \arrow[d, "\partial_2^*"'] & H \arrow[d, "\partial_2"] \\
\phi^*(N) \arrow[r, "\phi'"] \arrow[d, "\partial_1^*"'] & N \arrow[d, "\partial_1"] \\
P \arrow[r, "\phi"'] & Q
\end{tikzcd}$$

is a morphism of 2-crossed modules. (See appendix.)

Suppose that

$$(f_2, f_1, \phi) : \{B_2, B_1, P, \partial_2', \partial_1'\} \to \{H, N, Q, \partial_2, \partial_1\}$$

is any 2-crossed modules morphism

$$\begin{tikzcd}
B_2 \arrow[r, "\partial_2'"] \arrow[d, "f_2"'] & B_1 \arrow[r, "\partial_1'"] \arrow[d, "f_1"'] & P \arrow[d, "\phi"'] \\
H \arrow[r, "\partial_2"'] & N \arrow[r, "\partial_1"'] & Q.
\end{tikzcd}$$

Then we will show that there is a unique 2-crossed modules morphism

$$(f_2^*, f_1^*, id_P) : \{B_2, B_1, P, \partial_2', \partial_1'\} \to \{H, \phi^*(N), P, \partial_2^*, \partial_1^*\}$$

$$\begin{tikzcd}
B_2 \arrow[r, "\partial_2'"] \arrow[d, "f_2^*"'] & B_1 \arrow[r, "\partial_1'"] \arrow[d, "f_1^*"'] & P \arrow[d, equal, "id_P"'] \\
H \arrow[r, "\partial_2^*"'] & \phi^*(N) \arrow[r, "\partial_1^*"'] & P
\end{tikzcd}$$

where $f_2^*(b_2) = f_2(b_2)$ and $f_1^*(b_1) = (f_1(b_1), \partial_1'(b_1))$ which is an element in $\phi^*(N)$. First let us check that (f_2^*, f_1^*, id_P) is a 2-crossed modules morphism. For $b_1, b_1' \in B_1, b_2 \in B_2, p \in P$

$$\begin{aligned}
{}^{id_P(p)} f_2^*(b_2) &= {}^{p} f_2(b_2) \\
&= {}^{\phi(p)} f_2(b_2) \\
&= f_2({}^{p} b_2) \\
&= f_2^*({}^{p} b_2).
\end{aligned}$$

Similarly ${}^{id_P(p)}f_1^*(b_1) = f_1^*({}^{p}b_1)$, also above diagram is commutative and

$$\begin{array}{rcl} \{-,-\}(f_1^* \times f_1^*)(b_1, b_1') & = & \{-,-\}(f_1^*(b_1), f_1^*(b_1')) \\ & = & \{-,-\}((f_1(b_1), \partial_1'(b_1)), (f_1(b_1'), \partial_1'(b_1'))) \\ & = & \{f_1(b_1), f_1(b_1')\} \\ & = & \{-,-\}(f_1 \times f_1)(b_1, b_1') \\ & = & f_2\{-,-\}(b_1, b_1') \\ & = & f_2\{b_1, b_1'\} \\ & = & f_2^*\{b_1, b_1'\} \\ & = & f_2^*\{-,-\}(b_1, b_1'). \end{array}$$

Furthermore; the verification of the following equations are immediate.

$$id_H f_2^* = f_2 \quad \text{and} \quad \phi' f_1^* = f_1.$$

■

Thus we get a functor

$$\phi^* \colon \mathsf{X_2Mod}/Q \ \to \mathsf{X_2Mod}/P$$

which gives our pullback 2-crossed module.

3.1 Example of Pullback Two-Crossed Modules

Given 2-crossed module $\{\{1\}, G, Q, 1, i\}$ where i is an inclusion of a normal subgroup and a morphism $\phi : P \to Q$ of groups. The pullback 2-crossed module is

$$\begin{array}{rcl} \phi^*\{\{1\}, G, Q, 1, i\} & = & \{\{1\}, \phi^*(G), P, \partial_2^*, \partial_1^*\} \\ & = & \left\{\{1\}, \phi^{-1}(G), P, \partial_2^*, \partial_1^*\right\} \end{array}$$

as,

$$\begin{array}{rcl} \phi^*(G) & = & \{(g,p) \mid \phi(p) = i(g), g \in G, p \in P\} \\ & \cong & \{p \in P \mid \phi(p) = g\} = \phi^{-1}(G) \trianglelefteq P. \end{array}$$

The pullback diagram is

$$\begin{tikzcd} \{1\} \arrow[r, equal] \arrow[d, "\partial_2^*=1"'] & \{1\} \arrow[d] \\ \phi^{-1}(G) \arrow[r] \arrow[d, "\partial_1^*"'] & G \arrow[d, "i"] \\ P \arrow[r, "\phi"'] & Q. \end{tikzcd}$$

Particularly if $G = \{1\}$, then

$$\phi^*(\{1\}) \cong \{p \in P \mid \phi(p) = 1\} = \ker\phi \cong \{1\}$$

and so $\{\{1\},\{1\},P,\partial_2^*,\partial_1^*\}$ is a pullback 2-crossed module.

Also if ϕ is an isomorphism and $G=Q$, then $\phi^*(Q)=Q\times P$.

Similarly when we consider examples given in Section 1, the following diagrams are pullbacks.

$$\begin{tikzcd}
\{1\} \arrow[r, equal] \arrow[d, "\partial_2^*"'] & \{1\} \arrow[d] \\
\phi^*(Q) \arrow[r] \arrow[d, "\partial_1^*"'] & Q \arrow[d] \\
Aut(P) \arrow[r, "Aut\phi"'] & Aut(Q)
\end{tikzcd}
\qquad
\begin{tikzcd}
\{1\} \arrow[r, equal] \arrow[d, "\partial_2^*"'] & \{1\} \arrow[d] \\
N \times Ker\phi \arrow[r] \arrow[d, "\partial_1^*"'] & N \arrow[d, "0"] \\
P \arrow[r, "\phi"'] & Q
\end{tikzcd}$$

4 Induced Two-Crossed Modules

In this section we introduce the notion of an induced 2-crossed module. The concept of induced is given for crossed modules by Brown-Higgins in [2]. We will extend it to induced 2-crossed modules.

Definition 9 *For any* 2*-crossed module* $L \overset{\partial_2}{\to} M \overset{\partial_1}{\to} P$ *and group morphism* $\phi : P \to Q$*, the induced* 2*-crossed module can be given by*

(i) a 2-crossed module $\phi_*\{L,M,P,\partial_2,\partial_1\} = \{\phi_*(L),\phi_*(M),Q,\partial_2,\partial_1\}$
(ii) given any morphism of 2-crossed modules

$$(f_2,f_1,\phi):\{L,M,P,\partial_2,\partial_1\}\to\{B_2,B_1,Q,\partial_2',\partial_1'\}$$

then there is a unique (f_{2*},f_{1*},id_Q) 2-crossed modules morphism that commutes the following diagram:

$$\begin{tikzcd}
(L,M,P,\partial_2,\partial_1) \arrow[d, "(f_2,f_1,\phi)"'] \arrow[dr, "(\phi'',\phi',\phi)"] & \\
(B_2,B_1,Q,\partial_2',\partial_1') & (\phi_*(L),\phi_*(M),Q,\partial_{2_*},\partial_{1_*}) \arrow[l, dashed, "(f_{2_*},f_{1_*},id_Q)"]
\end{tikzcd}$$

or more simply as

$$
\begin{tikzcd}
 & & B_2 \arrow[dd, "{\partial_2}'"] \\
L \arrow[r, "\phi''"'] \arrow[urr, bend left=10, "f_2"] \arrow[dd, "\partial_2"'] & \phi_*(L) \arrow[ur, dashed, "f_{2*}"'] \arrow[dd, "\partial_{2*}"] & \\
 & & B_1 \arrow[dd, "{\partial_1}'"] \\
M \arrow[r, "\phi'"'] \arrow[urr, bend left=10, "f_1"] \arrow[dd, "\partial_1"'] & \phi_*(M) \arrow[ur, dashed, "f_{1*}"'] \arrow[dd, "\partial_{1*}"] & \\
 & & Q \\
P \arrow[r, "\phi"'] \arrow[urr, bend left=10, "\phi"] & Q. \arrow[ur, equal, "id_Q"'] &
\end{tikzcd}
$$

The following result is an extention of Proposition 4 given by Brown-Higgins in [2].

Proposition 10 *Let* $L \overset{\partial_2}{\to} M \overset{\partial_1}{\to} P$ *be a* 2*-crossed module and* $\phi : P \to Q$ *be a morphism of groups. Then* $\phi_*(L) \overset{\partial_{2_*}}{\to} \phi_*(M) \overset{\partial_{1_*}}{\to} Q$ *is the induced* 2*-crossed module where* $\phi_*(M)$ *is generated as a group, by the set* $M \times Q$ *with defining relations*

$$
\begin{aligned}
&(m,q)(m',q) = (mm',q) \\
&({}^{p}m,q) = (m,q\phi(p))
\end{aligned}
$$

and $\phi_*(L)$ *is generated as a group, by the set* $L \times Q$ *with defining relations*

$$
\begin{aligned}
&(l,q)(l',q) = (ll',q) \\
&({}^{p}l,q) = (l,q\phi(p))
\end{aligned}
$$

for all $l,l' \in L$, $m,m',m'' \in M$ *and* $q,q',q'' \in Q$*. The morphism* $\partial_{2*} : \phi_*(L) \to \phi_*(M)$ *is given by* $\partial_{2*}(l,q) = (\partial_2 l, q)$ *the action of* $\phi_*(M)$ *on* $\phi_*(L)$ *by* ${}^{(m,q)}(l,q) = ({}^{m}l,q)$*, and the morphism* $\partial_{1*} : \phi_*(M) \to Q$ *is given by* $\partial_{1*}(m,q) = q\phi(\partial_1(m))q^{-1}$*, the action of* Q *on* $\phi_*(M)$ *and* $\phi_*(L)$ *respectively by* ${}^{q}(m,q') = (m,qq')$ *and* ${}^{q}(l,q') = (l,qq')$*.*

Proof. As $\partial_{1_*}\partial_{2_*}(l,q) = \partial_{1_*}(\partial_2 l,q) = q\phi(\partial_1\partial_2 l)q^{-1} = q\phi(1)q^{-1} = 1$,

$$
\phi_*(L) \overset{\partial_{2_*}}{\to} \phi_*(M) \overset{\partial_{1_*}}{\to} Q
$$

is a complex of groups. The Peiffer lifting

$$
\{-,-\} : \phi_*(M) \times \phi_*(M) \to \phi_*(L)
$$

is given by $\{(m,q),(m',q)\} = (\{m,m'\},q)$.

PL1:

$$\begin{aligned}
(m,q)(m',q)(m,q)^{-1}\left({}^{\partial_{1_*}(m,q)}(m',q)^{-1}\right) &= \left(mm'm^{-1},q\right)\left({}^{\partial_{1*}(m,q)}\left(m'^{-1},q\right)\right)\\
&= \left(mm'm^{-1},q\right)\left({m'}^{-1},q\phi\partial_1(m)q^{-1}q\right)\\
&= \left(mm'm^{-1},q\right)\left(m'^{-1},q\phi\partial_1(m)\right)\\
&= \left(mm'm^{-1},q\right)\ \left({}^{\partial_1(m)}{m'}^{-1},q\right)\\
&= \left(mm'm^{-1}\ ({}^{\partial_1(m)}m'^{-1}),q\right)\\
&= \left(\partial_2\{m,m'\},q\right)\\
&= \partial_{2_*}\left(\{m,m'\},q\right)\\
&= \partial_{2*}\{(m,q),(m',q)\}.
\end{aligned}$$

PL2:

$$\begin{aligned}
\{\partial_{2_*}(l,q),\partial_{2_*}(l',q)\} &= \{(\partial_2 l,q),(\partial_2 l',q)\}\\
&= (\{\partial_2 l,\partial_2 l'\},q)\\
&= \left(ll'l^{-1}l'^{-1},q\right)\\
&= (l,q)(l',q)\left(l^{-1},q\right)\left(l'^{-1},q\right)\\
&= (l,q)(l',q)(l,q)^{-1}(l',q)^{-1}\\
&= [(l,q),(l',q)].
\end{aligned}$$

The rest of axioms of 2-crossed module is given in appendix.

(ii)

$$(\phi'',\phi',\phi):\{L,M,P,\partial_2,\partial_1\}\to\{\phi_*(L),\phi_*(M),Q,\partial_{2_*},\partial_{1_*}\}$$

or diagrammatically,

$$\begin{tikzcd}
L \arrow[r, "\phi''"] \arrow[d, "\partial_2"'] & \phi_*(L) \arrow[d, "\partial_{2_*}"] \\
M \arrow[r, "\phi'"] \arrow[d, "\partial_1"'] & \phi_*(M) \arrow[d, "\partial_{1_*}"] \\
P \arrow[r, "\phi"] & Q
\end{tikzcd}$$

is a morphism of 2-crossed modules. (See appendix.)

Suppose that

$$(f_2,f_1,\phi):\{L,M,P,\partial_2,\partial_1\}\to\{B_2,B_1,Q,\partial_2',\partial_1'\}$$

is any 2-crossed modules morphism. Then we will show that there is a 2-crossed modules morphism

$$(f_{2_*},f_{1_*},id_Q):\{\phi_*(L),\phi_*(M),Q,\partial_{2_*},\partial_{1_*}\}\to\{B_2,B_1,Q,\partial_2',\partial_1'\}$$

$$\begin{tikzcd}
\phi_*(L) \arrow[r, "\partial_{2*}"] \arrow[d, "f_{2*}"'] & \phi_*(M) \arrow[r, "\partial_{1*}"] \arrow[d, "f_{1*}"'] & Q \arrow[d, equal, "id_Q"'] \\
B_2 \arrow[r, "\partial_2'"'] & B_1 \arrow[r, "\partial_1'"'] & Q.
\end{tikzcd}$$

First we will check that (f_{2_*}, f_{1_*}, id_Q) is a 2-crossed modules morphism. We can see this easily as follows:

$$\begin{aligned} f_{2_*}\left({}^{q}(l, q')\right) &= f_{2_*}(l, qq') \\ &= {}^{qq'}f_2(l) \\ &= {}^{q}\left({}^{q'}f_2(l)\right) \\ &= {}^{q}\left(f_{2_*}(l, q')\right). \end{aligned}$$

Similarly $f_{1_*}\left({}^{q}(m, q')\right) = {}^{q}f_{1_*}(m, q')$,

$$\begin{aligned} \left(f_{1_*}\partial_{2*}\right)(l, q) &= f_{1_*}(\partial_2 l, q) \\ &= {}^{q}\left(f_1(\partial_2 l)\right) \\ &= {}^{q}\left(\left(\partial_2' f_2\right)(l)\right) \\ &= \partial_2'\left({}^{q}(f_2 l)\right) \\ &= \partial_2'\left(f_{2*}(l, q)\right) \\ &= \left(\partial_2' f_{2*}\right)(l, q) \end{aligned}$$

and $\partial_1' f_{1*} = id_Q \partial_{1_*}$ for $(m, q) \in \phi_*(M), (l, q) \in \phi_*(L), q \in Q$ and

$$f_{2*}\{-,-\} = \{-,-\}\left(f_{1_*} \times f_{1_*}\right).$$

■

Next if $\phi : P \longrightarrow Q$ is an epimorphism, the induced 2-crossed module has a simpler description.

Proposition 11 *Let $L \overset{\partial_2}{\to} M \to P$ is a 2-crossed module, $\phi : P \to Q$ is an epimorphism with Kerϕ=K. Then*

$$\phi_*(L) \cong L/[K, L] \text{ and } \phi_*(M) \cong M/[K, M],$$

where $[K, L]$ denotes the subgroup of L generated by $\{{}^{k}ll^{-1} \mid k \in K, l \in L\}$ and $[K, M]$ denotes the subgroup of M generated by $\{{}^{k}mm^{-1} \mid k \in K, m \in M\}$.

Proof. As $\phi : P \longrightarrow Q$ is an epimorphism, $Q \cong P/K$. Since Q acts on $L/[K, L]$ and $M/[K, M]$, K acts trivially on $L/[K, L]$ and $M/[K, M]$, $Q \cong P/K$ acts on $L/[K, L]$ by ${}^{q}(l[K, L]) = {}^{pK}(l[K, L]) = ({}^{p}l)[K, L]$ and $M/[K, M]$ by ${}^{q}(m[K, M]) = {}^{pK}(m[K, M]) = ({}^{p}m)[K, M]$ respectively.

$$L/[K, L] \overset{\partial_{2*}}{\to} M/[K, M] \overset{\partial_{1*}}{\to} Q$$

is a 2-crossed module where $\partial_{2*}(l[K, L]) = \partial_2(l)[K, M], \partial_{1*}(m[K, M]) = \partial_1(m)K$, the action of $M/[K, M]$ on $L/[K, L]$ by ${}^{m[K,M]}(l[K, L]) = ({}^{m}l)[K, L]$. As

$$\partial_{1_*}\partial_{2_*}(l[K, L]) = \partial_{1_*}\left(\partial_{2_*}(l[K, L])\right) = \partial_{1_*}\left(\partial_2(l)[K, L]\right) = \partial_1\left(\partial_2(l)\right)K = 1K \cong 1_Q,$$

$L/[K, L] \overset{\partial_{2*}}{\to} M/[K, M] \overset{\partial_{1*}}{\to} Q$ is a complex of groups.

The Peiffer lifting

$$M/[K,M] \times M/[K,M] \to L/[K,L]$$

given by $\{m[K,M], m'[K,M]\} = \{m, m'\}[K,L]$.

PL1:

$$\begin{aligned}
\partial_{2_*}\{m[K,M], m'[K,M]\} &= \partial_{2_*}\{m,m'\}[K,L] \\
&= (\partial_2\{m,m'\})[K,M] \\
&= \left(mm'm^{-1}\left({}^{\partial_1 m}m'^{-1}\right)\right)[K,M] \\
&= \left(mm'm^{-1}\right)[K,M] \ \left({}^{\partial_1 m}m'^{-1}\right)[K,M] \\
&= m[M,K]m'[K,M]m^{-1}[K,M] \ \left({}^{\partial_1 m}m'^{-1}\right)[K,M] \\
&= m[K,M]m'[K,M]m^{-1}[K,M] \ {}^{(\partial_1 mK)}m'^{-1}[K,M] \\
&= m[K,M]m'[K,M]\left(m[K,M]\right)^{-1} \ {}^{\partial_{1_*}(m[K,M])}\left(m'[K,M]\right)^{-1}.
\end{aligned}$$

PL2:

$$\begin{aligned}
\{\partial_{2_*}(l[K,L]), \partial_{2_*}(l'[K,L])\} &= \{\partial_2(l)[K,M], \partial_2(l')[K,M]\} \\
&= \{\partial_2(l), \partial_2(l')\}[K,L] \\
&= [l,l'][K,L] \\
&= \left(ll'l^{-1}l'^{-1}\right)[K,L] \\
&= (l[K,L])(l'[K,L])\left(l^{-1}[K,L]\right)\left(l'^{-1}[K,L]\right) \\
&= (l[K,L])(l'[K,L])(l[K,L])^{-1}(l'[K,L])^{-1} \\
&= [l[K,L], l'[K,L]].
\end{aligned}$$

The rest of axioms of 2-crossed module is given in appendix.

$$(\phi'', \phi', \phi) : \{L, M, P, \partial_2, \partial_1\} \longrightarrow \{L/[K,L], M/[K,M], Q, \partial_{2*}, \partial_{1_*}\}$$

or diagrammatically,

$$\begin{tikzcd}
L \arrow[r, "\phi''"] \arrow[d, "\partial_2"'] & L/[K,L] \arrow[d, "\partial_{2_*}"] \\
M \arrow[r, "\phi'"] \arrow[d, "\partial_1"'] & M/[K,M] \arrow[d, "\partial_{1_*}"] \\
P \arrow[r, "\phi"] & Q
\end{tikzcd}$$

is a morphism of 2-crossed modules.

Suppose that

$$(f_2, f_1, \phi) : \{L, M, P, \partial_2, \partial_1\} \longrightarrow \{B_2, B_1, Q, \partial_2', \partial_1'\}$$

is any 2-crossed modules morphism. Then we will show that there is a unique 2-crossed modules morphism

$$(f_{2*}, f_{1_*}, id_Q) : \{L/[K,L], M/[K,M], Q, \partial_{2*}, \partial_{1_*}\} \longrightarrow \{B_2, B_1, Q, \partial_2', \partial_1'\}$$

$$\begin{tikzcd}
L/[K,L] \arrow[r, "\partial_{2*}"] \arrow[d, "f_{2*}"'] & M/[K,M] \arrow[r, "\partial_{1*}"] \arrow[d, "f_{1*}"'] & Q \arrow[d, equal, "id_Q"'] \\
B_2 \arrow[r, "\partial_2'"'] & B_1 \arrow[r, "\partial_1'"'] & Q
\end{tikzcd}$$

where $f_{2*}(l[K,L]) = f_2(l)$ and $f_{1*}(m[K,M]) = f_1(m)$. Since

$$f_2({}^k l l^{-1}) = f_2({}^k l) f_2(l^{-1}) = f_2({}^k l) f_2(l)^{-1} = {}^{\phi(k)} f_2(l) f_2(l)^{-1} = {}^1 f_2(l) f_2(l)^{-1} = 1_{B_2},$$

$f_2([K,L]) = 1_{B_2}$ and similarly $f_1([K,M]) = 1_{B_1}$, thus f_{2*} and f_{1*} are well defined.

First let us check that (f_{2*}, f_{1_*}, id_Q) is a 2-crossed modules morphism. For $l[K,L] \in L/[K,L], m[K,M] \in M/[K,M]$ and $q \in Q$,

$$\begin{aligned}
f_{2_*}\left({}^q(l[K,L])\right) &= f_{2_*}\left({}^{pK}(l[K,L])\right) \\
&= f_{2_*}(({}^p l)[K,L]) \\
&= f_2({}^p l) \\
&= {}^{\phi(p)} f_2(l) \\
&= {}^{pK} f_{2_*}(l[K,L]) \\
&= {}^q\, f_{2_*}(l[K,L]).
\end{aligned}$$

Similarly $f_{1_*}\left({}^q(m[K,M])\right) = {}^q f_{1_*}(m[K,M])$,

$$\begin{aligned}
f_{1_*}\partial_{2*}(l[K,L]) &= f_{1_*}(\partial_2\,(l)\,[K,M]) \\
&= f_1\,(\partial_2\,(l)) \\
&= \partial_2'\,(f_2\,(l)) \\
&= \partial_2' f_{2_*}\,(l[K,L])
\end{aligned}$$

and $\partial_1' f_{1_*} = id_Q \partial_{1_*}$ and

$$\begin{aligned}
f_{2_*}\{-,-\}\,(m[K,M], m'[K,M]) &= f_{2_*}\{m[K,M], m'[K,M]\} \\
&= f_{2_*}\,(\{m,m'\}[K,L]) \\
&= f_2\{m,m'\} \\
&= f_2\{-,-\}(m,m') \\
&= \{-,-\}\,(f_1 \times f_1)\,(m,m') \\
&= \{f_1(m), f_1(m')\} \\
&= \{f_{1_*}(m[K,M]), f_{1_*}(m'[K,M])\} \\
&= \{-,-\}\,(f_{1_*} \times f_{1_*})\,(m[K,M], m'[K,M])\,.
\end{aligned}$$

So (f_{2*}, f_{1_*}, id_Q) is a morphism of 2-crossed modules. Furthermore; following equations are verified:

$$f_{2*}\phi'' = f_2 \text{ and } f_{1*}\phi' = f_1.$$

So given any morphism of 2-crossed modules

$$(f_2, f_1, \phi) : \{L, M, P, \partial_2, \partial_1\} \to \{B_2, B_1, Q, \partial_2', \partial_1'\}\,,$$

then there is a unique (f_{2*}, f_{1_*}, id_Q) 2-crossed modules morphism that commutes the following diagram:

$$\begin{tikzcd}
(L, M, P, \partial_2, \partial_1) \arrow[d, "(f_2, f_1, \phi)"'] \arrow[dr, "(\phi'', \phi', \phi)"] & \\
(B_2, B_1, Q, \partial_2', \partial_1') & (L/[K, L], M/[K, M], Q, \partial_{2_*}, \partial_{1_*}) \arrow[l, dashed, "(f_{2_*}, f_{1_*}, id_Q)"]
\end{tikzcd}$$

or more simply as

$$\begin{tikzcd}
 & & B_2 \arrow[dd, "\partial_2'"] \\
L \arrow[urr, bend left, "f_2"] \arrow[r, "\phi''"'] \arrow[dd, "\partial_2"'] & L/[K, L] \arrow[ur, dashed, "f_{2*}"'] \arrow[dd, "\partial_{2*}"] & \\
 & & B_1 \arrow[dd, "\partial_1'"] \\
M \arrow[urr, bend left, "f_1"] \arrow[r, "\phi'"'] \arrow[dd, "\partial_1"'] & M/[K, M] \arrow[ur, dashed, "f_{1*}"'] \arrow[dd, "\partial_{1*}"] & \\
 & & Q \arrow[dl, equal, "id_Q"] \\
P \arrow[urr, bend left, "\phi"] \arrow[r, "\phi"'] & Q. &
\end{tikzcd}$$

■

Corollary 12 *Let be any 2-crossed module $L \overset{\partial_2}{\to} M \overset{\partial_1}{\to} P$ and $\phi : P \to Q$ morphism of groups. Let $\phi_*(M)$ be induced pre-crossed module of $M \overset{\partial_1}{\to} P$ with ϕ and $\phi_*(L)$ be induced crossed module of $L \overset{\partial_2}{\to} M$ with $\phi' : M \to \phi_*(M)$. Then $\{\phi_*(L), \phi_*(M), Q, \partial_2, \partial_1\}$ is isomorphic to induced 2-crossed module of $L \overset{\partial_2}{\to} M \overset{\partial_1}{\to} P$ with ϕ.*

Proposition 13 *If $\phi : P \to Q$ is an injection and $L \overset{\partial_2}{\to} M \overset{\partial_1}{\to} P$ is a 2-crossed module, let T be a left transversal of $\phi(P)$ in H, and let B be the free product of groups L_T $(t \in T)$ each isomorphic with L by an isomorphism $l \mapsto l_t$ $(l \in L)$ and C be the free product of groups M_T $(t \in T)$ each isomorphic with by M by an isomorphism $m \mapsto m_t$ $(m \in M)$. Let $q \in Q$ act on B by the rule ${}^{q}(l_t) = ({}^{p}l)_u$ and similarly $q \in Q$ act on C by the rule ${}^{q}(m_t) = ({}^{p}m)_u$, where $p \in P, u \in T$, and $qt = u\phi(p)$. Let*

$$\begin{array}{llcll}
\gamma: & B \to C & \quad \textit{and} \quad & \delta: & C \to Q \\
 & l_t \mapsto \partial_2(l)_t & & & m_t \mapsto t(\phi\partial_1 m)t^{-1}
\end{array}$$

and the action of C on B by ${}^{(m_t)}(l_t) = ({}^{m}l)_t$. Then

$$\phi_*(L) = B \text{ and } \phi_*(M) = C$$

and the Peiffer lifting $C \times C \to B$ is given by $\{m_t, m'_t\} = \{m, m'\}_t$.

Remark 14 *Since any $\phi : P \to Q$ is the composite of a surjection and an injection, an alternative description of the general $\phi_*(L) \to \phi_*(M) \to Q$ can be obtained by a combination of the two constructions of Proposition 11 and Proposition 13.*

Now consider an arbitrary push-out square

$$\begin{tikzcd}
\{L_0, M_0, P_0, \partial_2, \partial_1\} \arrow[r] \arrow[d] & \{L_1, M_1, P_1, \partial_2, \partial_1\} \arrow[d] \\
\{L_2, M_2, P_2, \partial_2, \partial_1\} \arrow[r] & \{L, M, P, \partial_2, \partial_1\}
\end{tikzcd} \tag{1}$$

of 2-crossed modules. In order to describe $\{L, M, P, \partial_2, \partial_1\}$, we first note that P is the push-out of the group morphisms $P_1 \leftarrow P_0 \to P_2$. (This is because the functor

$$\{L, M, P, \partial_2, \partial_1\} \mapsto (M/\backsim, P, \partial_1)$$

from 2-crossed modules to crossed modules has a right adjoint $(N, P, \partial) \mapsto \{1, N, P, 1, \partial\}$ and the forgetful functor $(M/\backsim, P, \partial_1) \mapsto P$) from crossed module to group where $\backsim$ is the normal closure in M of the elements $\left({}^{\partial_1 m}m'\right) mm'^{-1}m^{-1}$ for $m, m' \in M$ has a right adjoint $P \mapsto (P, P, Id)$.) The morphisms $\phi_i : P_i \to P$ $(i = 0, 1, 2)$ in (1) can be used to form induced 2-crossed Q-modules $B_i = (\phi_i)_* L_i$ and $C_i = (\phi_i)_* M_i$. Clearly $\{L, M, P, \partial_2, \partial_1\}$ is the push-out in $\mathsf{X_2Mod}/P$ of the resulting P-morphisms

$$(B_1 \to C_1 \to P) \longleftarrow (B_0 \to C_0 \to P) \longrightarrow (B_2 \to C_2 \to P)$$

can be described as follows.

Proposition 15 *Let $(B_i \to C_i \to P)$ be a 2-crossed P-module for $i = 0, 1, 2$ and let $(L \to M \to P)$ be the push-out in $\mathsf{X_2Mod}/P$ of P-morphisms*

$$(B_1 \to C_1 \to P) \xleftarrow{(\alpha_1, \beta_1, Id)} (B_0 \to C_0 \to P) \xrightarrow{(\alpha_2, \beta_2, Id)} (B_2 \to C_2 \to P)$$

Let $(B \to M)$ be the push-out of (α_1, β_1) and (α_2, β_2) in XMod, equipped with the induced morphism $B \xrightarrow{\mu} C \xrightarrow{\nu} P$, the lifting

$$\{-,-\} : C \times C \to B$$

and the induced action of P on B and C. Then $L = B/S$, where S is the normal closure in B of the elements

$$\begin{gathered}\{\mu(b), \mu(b')\}[b, b']^{-1} \\ \{c, c'c''\}\{c, c'\}^{-1}\left({}^{cc'c^{-1}}\{c, c''\}\right)^{-1} \\ \{cc', c''\}\left({}^{\nu(c)}\{c', c''\}\right)^{-1}\left\{c, c'c''c'^{-1}\right\}^{-1} \\ \{\mu(b), c\}\left({}^{c}b^{-1}\right)^{-1}b^{-1} \\ \{c, \mu(b)\}\left({}^{\nu(c)}b^{-1}\right)^{-1}\left({}^{c}b\right)^{-1} \\ {}^{p}\{c, c'\}\{{}^{p}c, {}^{p}c'\}^{-1}\end{gathered}$$

and $M = C/R$, where R is the normal closure in C of the elements

$$\mu\{c, c'\}\left({}^{\nu(c)}c'^{-1}\right)^{-1}cc'^{-1}c^{-1}$$

for $b, b' \in B, c, c', c'' \in C$ and $p \in P$.

In the case when $\{L_2, M_2, P_2, \partial_2, \partial_1\}$ is the trivial 2-crossed module $\{1, 1, 1, id, id\}$ the push-out $\{L, M, P, \partial_2, \partial_1\}$ in $(*)$ is the cokernel of the morphism

$$\{L_0, M_0, P_0, \partial_2, \partial_1\} \to \{L_1, M_1, P_1, \partial_2, \partial_1\}$$

Cokernels can be described as follows.

Proposition 16 *$Q/\bar{P}$ is the push-out of the group morphisms $1 \leftarrow P \to Q$. Let $\left\{A_*, G_*, Q/\bar{P}, \partial_2, \partial_1\right\}$ be the induced from $\{A, G, P, \partial_2, \partial_1\}$ by $P \to Q/\bar{P}$. If $\left\{1, 1, Q/\bar{P}, id, \partial_1\right\}$ and*

$$\left\{B/\left[\bar{P}, B\right], H/\left[\bar{P}, H\right], Q/\bar{P}, \partial_2, \partial_1\right\}$$

are the induced from $\{1, 1, 1, id, id\}$ and $\{B, H, Q, \partial_2, \partial_1\}$ by $1 \to Q/\bar{P}$ and the epimorphism $Q \to Q/\bar{P}$ then the cokernel of a morphism

$$(\beta, \lambda, \phi) : \{A, G, P, \partial_2, \partial_1\} \to \{B, H, Q, \partial_2, \partial_1\}$$

is $\left\{coker\ (\beta_*, \lambda_*), Q/\bar{P}, \partial_2, \partial_1\right\}$ where (β_*, λ_*) is a morphism of

$$(A_*, G_*) \to \left(B/\left[\bar{P}, B\right], H/\left[\bar{P}, H\right]\right).$$

5 Appendix

The proof of proposition 8

PL3: *a)*
$$\begin{aligned} & {}^{(n,p)\left(n',p'\right)(n,p)^{-1}}\{(n,p), (n'', p'')\}\{(n,p), (n', p')\} \\ = \; & {}^{\left(nn'n^{-1}, pp'p^{-1}\right)}\{n, n''\}\{n, n'\} \\ = \; & {}^{nn'n^{-1}}\{n, n''\}\{n, n'\} \\ = \; & \{n, n'n''\} \\ = \; & \{(n,p), (n'n'', p'p'')\} \\ = \; & \{(n,p), (n', p')(n'', p'')\}. \end{aligned}$$

$$\begin{aligned}
&b)\ \left\{(n,p)\,,(n',p')\,(n'',p'')\,(n',p')^{-1}\right\}\left({}^{\partial_1^*(n,p)}\left\{(n',p')\,,(n'',p'')\right\}\right)\\
&=\left\{(n,p)\,,\left(n'n''n'^{-1},p'p''p'^{-1}\right)\right\}^p\left\{(n',p')\,,(n'',p'')\right\}\\
&=\left\{n,n'n''n'^{-1}\right\}^p\left\{n',n''\right\}\\
&=\left\{n,n'n''n'^{-1}\right\}^{\phi(p)}\left\{n',n''\right\}\\
&=\left\{n,n'n''n'^{-1}\right\}^{\partial_1(n)}\left\{n',n''\right\}\\
&=\left\{nn',n''\right\}\\
&=\left\{(nn',pp')\,,(n'',p'')\right\}\\
&=\left\{(n,p)\,(n',p')\,,(n'',p'')\right\}.
\end{aligned}$$

PL4:

$$\begin{aligned}
\left\{\partial_2^*h,(n,p)\right\}\left\{(n,p)\,,\partial_2^*h\right\} &= \left\{(\partial_2h,1)\,,(n,p)\right\}\left\{(n,p)\,,(\partial_2h,1)\right\}\\
&= \left\{\partial_2h,n\right\}\left\{n,\partial_2h\right\}\\
&= h\ {}^{\partial_1(n)}h^{-1}\\
&= h\ {}^{\phi(p)}h^{-1}\\
&= h\ {}^{p}h^{-1}\\
&= h\ {}^{\partial_1^*(n,p)}h^{-1}.
\end{aligned}$$

PL5:

$$\begin{aligned}
\left\{{}^{p''}(n,p)\,,{}^{p''}(n',p')\right\} &= \left\{\left({}^{\phi(p'')}n,p''p\,(p'')^{-1}\right)\left({}^{\phi(p'')}n',p''p'\,(p'')^{-1}\right)\right\}\\
&= \left\{{}^{\phi(p'')}n,{}^{\phi(p'')}n'\right\}\\
&= {}^{\phi(p'')}\left\{n,n'\right\}\\
&= {}^{p''}\left\{n,n'\right\}\\
&= {}^{p''}\left\{(n,p)\,,(n',p')\right\}.
\end{aligned}$$

$$i)\quad \begin{aligned} id_H\left({}^ph\right) &= {}^ph\\ &= {}^{\phi(p)}h\\ &= {}^{\phi(p)}id_H\left(h\right)\end{aligned}\qquad\text{and}\qquad \begin{aligned}\phi'\left({}^pn,p'\right) &= \phi'\left({}^{\phi(p)}n,pp'p^{-1}\right)\\ &= {}^{\phi(p)}n\\ &= {}^{\phi(p)}\phi'\left(n,p'\right).\end{aligned}$$

$$ii)\quad \begin{aligned}\phi'\left(\partial_2^*h\right) &= \phi'\left(\partial_2h,1\right)\\ &= \partial_2\left(h\right)\\ &= \partial_2\left(id_Hh\right)\end{aligned}\qquad\text{and}\qquad\begin{aligned}\partial_1\left(\phi'\left(n,p'\right)\right) &= \partial_1\left(n\right)\\ &= \phi\left(p'\right)\\ &= \phi\left(\partial_1^*\left(n,p'\right)\right).\end{aligned}$$

$$\begin{aligned}
\left\{-,-\right\}\left(\phi'\times\phi'\right)\left((n,p),(n',p')\right) &= \left\{-,-\right\}\left(\phi'(n,p),\phi'(n',p')\right)\\
&= \left\{-,-\right\}\left(n,n'\right)\\
&= \left\{n,n'\right\}\\
&= id_H\left(\left\{n,n'\right\}\right)\\
&= id_H\left(\left\{(n,p),(n',p')\right\}\right)\\
&= id_H\left\{-,-\right\}\left((n,p),(n',p')\right).
\end{aligned}$$

The proof of proposition 10:

PL3:

a)

$$\begin{aligned}
& \{(m,q),(m',q)(m'',q)\} \\
= & \{(m,q),(m'm'',q)\} \\
= & (\{m,m'm''\},q) \\
= & \left({}^{mm'm^{-1}}\{m,m''\}\{m,m'\},q\right) \\
= & \left({}^{mm'm^{-1}}\{m,m''\},q\right)(\{m,m'\},q) \\
= & {}^{(mm'm^{-1},q)}(\{m,m''\},q)(\{m,m'\},q) \\
= & {}^{(m,q)(m',q)(m,q)^{-1}}\{(m,q),(m'',q)\}\{(m,q),(m',q)\}.
\end{aligned}$$

b)

$$\begin{aligned}
& \{(m,q)(m',q),(m'',q)\} \\
= & \{(mm',q),(m'',q)\} \\
= & (\{mm',m''\},q) \\
= & \left(\left\{m,m'm''m'^{-1}\right\}^{\partial_1(m)}\{m',m''\},q\right) \\
= & \left(\left\{m,m'm''m'^{-1}\right\},q\right)(\{m',m''\},q\phi\partial_1(m)) \\
= & \left(\left\{m,m'm''m'^{-1}\right\},q\right)\left(\{m',m''\},q\phi\partial_1(m)q^{-1}q\right) \\
= & \left(\left\{m,m'm''m'^{-1}\right\},q\right)(\{m',m''\},\partial_{1_*}(m,q)q) \\
= & \left(\left\{m,m'm''m'^{-1}\right\},q\right)^{\partial_{1_*}(m,q)}(\{m',m''\},q) \\
= & \left\{(m,q),(m',q)(m'',q)(m',q)^{-1}\right\}^{\partial_{1_*}(m,q)}\{(m',q),(m'',q)\}.
\end{aligned}$$

PL4:

a)

$$\begin{aligned}
\{\partial_{2_*}(l,q),(m,q)\} &= \{(\partial_2 l,q),(m,q)\} \\
&= (\{\partial_2 l,m\},q) \\
&= \left(l^m l^{-1},q\right) \\
&= (l,q)\left({}^m l^{-1},q\right) \\
&= (l,q)^{(m,q)}(l,q)^{-1}.
\end{aligned}$$

b)

$$\begin{aligned}
\{(m,q),\partial_{2_*}(l,q)\} &= \{(m,q),(\partial_2 l,q)\} \\
&= (\{m,\partial_2 l\},q) \\
&= \left({}^m l\left({}^{\partial_1(m)}l^{-1}\right),q\right) \\
&= ({}^m l,q)\left({}^{\partial_1(m)}l^{-1},q\right) \\
&= ({}^m l,q)\ \left(l^{-1},q\phi\partial_1(m)\right) \\
&= ({}^m l,q)\ \left(l^{-1},q\phi\partial_1(m)q^{-1}q\right) \\
&= ({}^m l,q)\ \left(l^{-1},\partial_{1*}(m,q)\,q\right) \\
&= ({}^m l,q)^{\ \partial_{1*}(m,q)}\left(l^{-1},q\right) \\
&= {}^{(m,q)}(l,q)^{\ \partial_{1_*}(m,q)}(l,q)^{-1}.
\end{aligned}$$

PL5:

$$\begin{aligned}
{}^{q'}\{(m,q),(m',q)\} &= {}^{q'}(\{m,m'\},q) \\
&= (\{m,m'\},q'q) \\
&= \{(m,q'q),(m',q'q)\} \\
&= \left\{{}^{q'}(m,q),{}^{q'}(m',q)\right\}.
\end{aligned}$$

$$\begin{array}{rcl} \phi''({}^{p}l) &=& ({}^{p}l,1) \\ &=& (l,1\phi(p)) \\ &=& (l,\phi(p)1) \\ &=& {}^{\phi(p)}\,(l,1) \\ &=& {}^{\phi(p)}\phi''(l) \end{array} \quad \text{and} \quad \begin{array}{rcl} \phi'({}^{p}m) &=& ({}^{p}m,1) \\ &=& (m,1\phi(p)) \\ &=& (m,\phi(p)1) \\ &=& {}^{\phi(p)}\,(m,1) \\ &=& {}^{\phi(p)}\phi'(m). \end{array}$$

$$\begin{array}{rcl} \partial_{2*}\left(\phi''\left(l\right)\right) &=& \partial_{2*}\left(l,1\right) \\ &=& \left(\partial_2 l,1\right) \\ &=& \phi'\left(\partial_2 l\right) \end{array} \quad \text{and} \quad \begin{array}{rcl} \partial_{1_*}\left(\phi'\left(m\right)\right) &=& \partial_{1_*}\left(m,1\right) \\ &=& 1\phi\left(\partial_1(m)\right)1^{-1} \\ &=& \phi\left(\partial_1(m)\right). \end{array}$$

The proof of proposition 11:

PL3:

$a)$
$$\begin{array}{rl} & \{m[K,M],m'[K,M]m''[K,M]\} \\ = & \{m[K,M],\left(m'm''\right)[K,M]\} \\ = & \{m,m'm''\}\,[K,L] \\ = & \left({}^{mm'm^{-1}}\,\{m,m''\}\{m,m'\}\right)[K,L] \\ = & ({}^{mm'm^{-1}}\{m,m''\})[K,L](\{m,m'\})[K,L] \\ = & {}^{\left(mm'm^{-1}[K,M]\right)}(\{m,m''\}[K,L])(\{m,m'\})[K,L] \\ = & {}^{\left(mm'm^{-1}[K,M]\right)}\,\{m[K,M],m''[K,M]\}\,\{m[K,M],m'[K,M]\} \\ = & {}^{\left(m[K,M]m'[K,M](m[K,M])^{-1}\right)}\,\{m[K,M],m''[K,M]\}\,\{m[K,M],m'[K,M]\}\,. \end{array}$$

$b)$
$$\begin{array}{rl} & \{m[K,M]m'[K,M],m''[K,M]\} \\ = & \{\left(mm'\right)[K,M],m''[K,M]\} \\ = & \{mm',m''\}\,[K,L] \\ = & \left(\left\{m,m'm''m'^{-1}\right\}\ {}^{\partial_1(m)}\,\{m',m''\}\right)[K,L] \\ = & \left(\left\{m,m'm''m'^{-1}\right\}[K,L]\right)\left({}^{\partial_1(m)}\,\{m',m''\}\,[K,L]\right) \\ = & \left\{m[K,M],\left(m'm''m'^{-1}\right)[K,M]\right\}\left({}^{\partial_1(m)}\,\{m',m''\}\,[K,L]\right) \\ = & \left\{m[K,M],m'[K,M]m''[K,M]\,(m'[K,M])^{-1}\right\}\left({}^{\partial_1(m)}\,\{m',m''\}\,[K,L]\right) \\ = & \left\{m[K,M],m'[K,M]m''[K,M]\,(m'[K,M])^{-1}\right\}^{\partial_1(m)K}(\{m',m''\}\,[K,L]) \\ = & \left\{m[K,M],m'[K,M]m''[K,M]\,(m'[K,M])^{-1}\right\}^{\partial_{1_*}(m[K,M])}(\{m',m''\}\,[K,L])\,. \end{array}$$

PL4:

$a)$
$$\begin{array}{rcl} \{\partial_{2*}(l[K,L]),m[K,M]\} &=& \{\partial_2\left(l\right)[K,M]),m[K,M]\} \\ &=& \{\partial_2\left(l\right),m\}\,[K,L] \\ &=& \left(l^{m}l^{-1}\right)[K,L] \\ &=& l[K,L]\left({}^{m}l^{-1}[K,L]\right) \\ &=& l[K,L]^{m[K,M]}\left(l^{-1}[K,L]\right) \\ &=& l[K,L]^{m[K,M]}\left(l[K,L]\right)^{-1} \end{array}$$

$$
\begin{aligned}
b)\quad \{m[K,M],\partial_{2*}(l[K,L])\} &= \{m[K,M],\partial_2(l)[K,M])\} \\
&= \{m,\partial_2(l)\}[K,L] \\
&= \left({}^{m}l^{\partial_1(m)}l^{-1}\right)[K,L] \\
&= \left({}^{m}l\right)[K,L]\ \left({}^{\partial_1(m)}l^{-1}\right)[K,L] \\
&= {}^{m[K,M]}\left(l[K,L]\right)\ {}^{\partial_1(m)K}\left(l^{-1}[K,L]\right) \\
&= {}^{m[K,M]}\left(l[K,L]\right)\ {}^{\partial_{1*}(m[K,M])}\left(l[K,L]\right)^{-1}.
\end{aligned}
$$

PL5:

$$
\begin{aligned}
{}^{q}\{m[K,M],m'[K,M]\} &= {}^{pK}\{m[K,M],m'[K,M]\} \\
&= {}^{pK}\left(\{m,m'\}[K,L]\right) \\
&= \left({}^{p}\{m,m'\}\right)[K,L] \\
&= \{{}^{p}m,{}^{p}m'\}[K,L] \\
&= \{({}^{p}m)[K,M],({}^{p}m')[K,M]\} \\
&= \left\{{}^{pK}(m[K,M]),{}^{pK}(m'[K,M])\right\} \\
&= \left\{{}^{q}(m[K,M]),{}^{q}(m'[K,M])\right\}.
\end{aligned}
$$

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